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SIMPLIFYING PARTIAL DIFFERENTIAL EQUATIONS BY FEEDBACK

L.R. Hunt[1] and Ramiro Villarreal

Programs in Mathematical Sciences

The University of Texas at Dallas

P.O. Box 830688

Richardson, Texas 75083-0688

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ABSTRACT

Given a partial differential equation, analysts often ask for conditions under which the equation takes a simplified form. For example, we consider coordinate changes that make the highest order part constant coefficient or Lie transformations that reduce the number of variables. However, if we view a partial differential equation as an input-output system, then the prospect of a simplifying feedback arises.

For systems of nonlinear control ordinary differential equations, much research and many applications have unfolded concerning the feedback equivalence with controllable linear systems. In a sense the well understood linear control system is an ideal model that we use for design. We consider second order linear partial differential equations with variable coefficients and propose for the parallel to the linear control system equations due to Kolmogorov. In these equations the second order spatial part is constant coefficient and the first order spatial part has linearly varying coefficients. We then examine the problem of feedback equivalence after introducing an interesting type of feedback.

Keywords: partial differential equations, partial differential operators, feedback, transformations, canonical forms.

SIMPLIFYING PARTIAL DIFFERENTIAL

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I. Introduction

We consider partial differential equations of the forms

$$(1)\quad -\frac{\partial u}{\partial t} + \sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} + C(x)u = f$$

$$(2)\quad -\frac{\partial^2 u}{\partial t^2} + \sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} + C(x)u = f,$$

where the coefficients are C^∞ functions and f is the input. Studies of canonical coordinates in p.d.e's involve necessary and sufficient conditions that coordinate changes exist for $(x_1, x_2, \ldots, x_n)$ to transform the coefficients of the second order terms to constants (see [G], [CH], [C], [F]). We shall expand our transformations to include appropriate feedback operations and ask for (local) transformations that move (1) and (2) to

$$(3)\quad -\frac{\partial u}{\partial t} + \sum_{j,k=1}^{n} a_{jk}\cancel{(x)} \frac{\partial^2 u}{\partial y_j \partial y_k} + \sum_{j,k=1}^{n} b_{jk}\, y_j \frac{\partial u}{\partial y_k} = \tilde{f}$$

$$(4)\quad -\frac{\partial^2 u}{\partial t^2} + \sum_{j,k=1}^{n} a_{jk}\cancel{(x)} \frac{\partial^2 u}{\partial y_j \partial y_k} + \sum_{j,k=1}^{n} b_{jk}\, y_j \frac{\partial u}{\partial y_k} = \bar{f}$$

respectively, where $(y_1, y_2, \ldots, y_n)$ are the new spatial variables, the a_{jk} and b_{jk} are constants, and $\tilde{f}$ and $\bar{f}$ denote new inputs. We restrict our attention to the case that the matrices $(A_{jk}(x))$ in (1) and (2) and $A = (a_{jk})$ in (3) and (4) are symmetric, positive semidefinite, and have constant rank m. We also require that if $B = (b_{jk})$ in (3) and (4), then the matrix $(A, BA, B^2A, \ldots, B^{n-1}A)$ has rank n.

For the problems of interest we need only examine the spatial partial differential equations

$$(5) \qquad \sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} + C(x)u = f$$

$$(6) \qquad \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial y_j \partial y_k} + \sum_{j,k=1}^{n} b_{jk}\, y_j \frac{\partial u}{\partial y_k} = \bar{f}$$

where $(A_{jk}(x))$, $A = (a_{jk})$ and $B = (b_{jk})$ are as above. We seek necessary and sufficient conditions to transform (in some proper sense) equation (5) to equation (6). If $C(x) = 0$ and only nonsingular coordinates changes on $\mathbb{R}^n$ are allowed, the results are known [HV]. We introduce a feedback operation to enrich our transformations. This parallels work on moving nonlinear control systems of ordinary differential equations to controllable linear systems by feedback transformations [MC1], [MC2], [B], [JR], [S], [S1], [HSM], [HS], [KIR], [SH]. Applications of these o.d.e. theories are numerous [MC1], [MC2], [BMHS], [SM2], [W1], [WM], [M], [MSH], [TBIC], [D1], [D2], [HH], [FKD], [KC], [AE], [K1], [SMPT].

The main thrust of this paper is to examine transformations of the p.d.e. (5) to the p.d.e. (6). However, we also show that equation (6) is to p.d.e.'s as the linear system

(7) $$\dot{y} = \frac{dy}{dt} = By + Av$$

is to o.d.e. control systems. Here the usual rolls of A and B in linear systems have been interchanged to fit our previously introduced notation, and v is an m dimensional control vector. Section 2 of this paper contains the linear theory discussion and an introduction to our linear feedback. Section 3 involves definitions and conditions under which equation (5) transforms to equation (6). In section 4 we discuss the special case of elliptic equations and examine the possibility of finite difference and finite element implementations if there are a finite number of point sensors and actuators.

II. Linear Systems and Feedback

We develop a parallel viewpoint for linear control p.d.e.'s like (6) that classical textbooks take for linear controllable o.d.e. systems. Consider

(8) $$\dot{y} = By + Av,$$

where $y \in \mathbb{R}^n$, $v=(v_1, v_2, \ldots, v_m)$, B is $n\times n$, and A is $n\times m$ with rank m. If the system is controllable (i.e. the matrix $(A, BA, B^2A, \ldots, B^{n-1}A)$ has rank n, then linear feedback $v = Ky$ can be used so that the eigenvalues of $\dot{y} = (B+AK)y$ are arbitraily placed with complex conjugates in pairs.

For the partial differential equation (6)

$$\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial y_j \partial y_k} + \sum_{j,k=1}^{n} b_{jk}\, y_j \frac{\partial u}{\partial y_k} = \bar{f}$$

we can write (see Hörmander [H])

$$\text{(9)} \qquad \sum_{j=1}^{m} Y_j^2\, u + Y_0 u = \bar{f},$$

where $Y_1, Y_2, \ldots, Y_m$ are linearly independent constant coefficient first order partial differential operators (vector fields) and Y_0 is a linearly varying vector field $\left[Y_0 = \sum_{j,k=1}^{n} b_{jk} y_j \frac{\partial}{\partial y_k}\right]$. The partial differential equation (6) (or (1) if time is included) is called an <u>equation of Kolmogorov type</u> or <u>Kolmogorov equation</u> if it satisfies the hypoellipticity condition mentioned below.

Hörmander indicates that our equation (5) can be written in the form

$$\text{(10)} \qquad \sum_{j=1}^{m} X_j^2 u + X_0 u + C(x)u = f,$$

where $X_0, X_1, X_2, \ldots, X_m$ are vector fields with $X_1, X_2, \ldots, X_m$ being linearly independent. He then derives necessary and sufficient conditions that equation (10) be <u>hypoelliptic</u> (i.e. C^∞ right hand side f implies C^∞ solution). For equation (6) these conditions for hypoellipticity is that the matrix $(A, BA, B^2A, \ldots, B^{n-1}A)$ has rank n.

Our problem of transforming equation (5) to equation (6) can be interpreted in terms of transforming (10) to (9), under the proper assumptions of hypoellipticity for all equations.

First we restrict our attention to the linear system (6). We want to derive canonical forms where the transformations considered involve nonsingular linear coordinate changes on $\mathbb{R}^n$ and appropriate linear feedback. Our linear feedback takes the form

$$\bar{f} = \sum_{j=1}^{m} k_j y Y_j u, \tag{11}$$

where each k_j is 1×n matrix of real constants. This means we can feedback a sum of terms involving a linear combination of y variables times a vector field applied to the solution u. Each Y_j must be one of the vector fields forming the principal part of the operator. This type of feedback does not disturb the principal symbol or the hypoelliptic assumption. We prove that such a feedback can arbitrarily place the eigenvalues of the B matrix in (6). An example is provided after the proof.

<u>Theorem 2.1</u>. Consider the partial differential equation (6)

$$\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial y_j \partial y_k} + \sum_{j,k=1}^{n} b_{jk}\, y_j \frac{\partial u}{\partial y_k} = \bar{f}$$

which can also be described by (9)

$$\sum_{j=1}^{m} Y_j^2\, u + Y_0 u = \bar{f}.$$

Assume that the matrix $(A, BA, B^2A, \ldots, B^{n-1}A)$ has rank n, where $A = (a_{jk})$ and $B = (b_{jk})$. Then the eigenvalues of the B matrix can be arbitrarily placed (with complex eigenvalues in conjugate pairs) by linear feedback.

<u>Proof</u>. We assume that the corresponding linear ordinary differential equation control system (8)

$$\dot{y} = By + Av$$

has Kronecker indices $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m$ and is in controllable canonical form. In these same coordinates B takes the form

$$
\left[\begin{array}{c|c|c|c}
0\;1\;0\cdots 0 & 0\;0\cdots 0 & & 0\cdots 0\\
0\;0\;1\;0\cdots 0 & 0\;0\cdots 0 & & 0\;0\cdots 0\\
\vdots & \vdots & \cdots & \vdots\\
\;\cdots * & *\;*\cdots * & & *\;*\cdots *\\ \hline
0\;0\cdots 0 & 0\;1\;0\cdots 0 & & 0\;0\cdots 0\\
0\;0\cdots 0 & 0\;0\;1\;0\cdots 0 & & 0\;0\cdots 0\\
\vdots & \vdots & \cdots & \vdots\\
\;\cdots * & *\;*\cdots * & & *\;*\cdots *\\ \hline
\vdots & \vdots & \ddots & \vdots\\ \hline
0\;0\cdots 0 & 0\;0\cdots 0 & & 0\;1\;0\cdots 0\\
0\;0\cdots 0 & 0\;0\cdots 0 & & 0\;0\;1\;0\cdots 0\\
\vdots & \vdots & \cdots & \vdots\\
\;\cdots * & *\;*\cdots * & & *\;*\cdots *
\end{array}\right]
\begin{array}{l}
\\ \\ \kappa_1 \\ \\ \\ \\ \kappa_2 \\ \\ \\ \\ \\ \kappa_m \\ \\
\end{array}
\tag{12}
$$

and $Y_1 = \dfrac{\partial}{\partial x_{\kappa_1}}$, $Y_2 = d_{21}\dfrac{\partial}{\partial x_{\kappa_1}} + \dfrac{\partial}{\partial x_{\kappa_2}}$, $\cdots$, $Y_m = \displaystyle\sum_{i=1}^{m-1} d_{mi}\dfrac{\partial}{\partial x_{\kappa_i}} + \dfrac{\partial}{\partial x_{\kappa_m}}$,

where $d_{21}, \cdots, d_{m1}, \cdots, d_{m,m-1}$ are real constants.

Suppose we have an arbitrary n^{th} degree polynomial $s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$ having real coefficients. For

$$\bar{f} = \sum_{j=1}^{m} k_j y Y_j u$$

we choose $k_1, k_2, \ldots, k_m$ to drive B to rational canonical form having last row $[-\alpha_n \; -\alpha_{n-1} \; \cdots \; -\alpha_1]$. The desired eigenvalue placement is now accomplished. □

Given a Kolmogorov equation (6) we have the associated controllable linear system $\dot{y} = By + Av$. The Kronecker indices $\kappa_1, \kappa_2, \ldots, \kappa_m$ of $\dot{y} = By + Av$ give us the Kolmogorov indices [HV], denoted by $\ell_1, \ell_2, \ldots, \ell_m$ of the Kolmogorov equation. These can be computed directly from (6) using the matrices A and B. We have $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m$ and $\ell_1 + \ell_2 + \cdots + \ell_m = n$.

Example 2.1 Consider the hypoelliptic control partial differential equation in $\mathbb{R}^2$

$$\sum_{j,k=1}^{2} a_{jk} \frac{\partial^2 u}{\partial y_j \partial y_k} + \sum_{j,k=1}^{2} b_{jk}\, y_j \frac{\partial u}{\partial y_k} = \bar{f} \tag{13}$$

where the symmetric, positive semidefinite matrix $A = (a_{jk})$ has rank 1. We want to use coordinate changes and linear feedback so that (13) becomes

$$\frac{\partial^2 u}{\partial y_2^2} + y_2 \frac{\partial u}{\partial y_1} = \bar{\bar{f}}. \tag{14}$$

Here y_1 and y_2 are our new coordinates and $\bar{\bar{f}}$ our new input. The associated differential equation

$$-\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial y_2^2} + y_2 \frac{\partial u}{\partial y_1} = \bar{\bar{f}} \tag{15}$$

was shown by Kolmogorov [K2] and Weber [W2] to have fundamental solution

$$(16)\qquad \frac{2\sqrt{3}}{\pi(t-\tau)} \quad \exp\left[-\frac{(y_2-y_2')^2}{4(t-\tau)} - 3\,\frac{\{y_1' - y_1 - 2^{-1}(t-\tau)(y_2'+y_2)\}^2}{(t-\tau)^2}\right].$$

The vector field version of (13) is

$$(17)\qquad Y_1^2 + Y_0 = \bar{f}.$$

We assume that coordinate changes have been made so that $Y_1 = \begin{bmatrix}0\\1\end{bmatrix}$ and $Y_0 = \begin{bmatrix}0 & 1\\ -\beta_2 & -\beta_1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$, where the characteristic polynomial of B is $s^2 + \beta_1 s + \beta_2$.

Adding $[\beta_2\ \ \beta_1]\begin{bmatrix}y_1\\y_2\end{bmatrix} Y_1 \left[\text{i.e. } \beta_2 y_1 \frac{\partial}{\partial y_1} + \beta_1 y_2 \frac{\partial}{\partial y_2}\right]$ to both sides of (17) and replacing $\bar{f} + [\beta_2\ \beta_1]\begin{bmatrix}y_1\\y_2\end{bmatrix}Y_1$ by $\bar{\bar{f}}$, we have the partial differential equation (14)

$$\frac{\partial^2 u}{\partial y_2^2} + y_2 \frac{\partial u}{\partial y_1} = \bar{\bar{f}}.$$

The diffusion equation (15) of Fokker-Planck type with zero right hand side is satisfied by a probability density of a system with 2 degrees of freedom. The hypoellipticity results of Hörmander are used by Elliott [E1], [E2] to prove smoothness for certain probability density functions.

We consider the effect of feedback of the form (11) on the partial differential equation (6). Following the arguments in [H] we take Fourier transforms with $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ being the transform variables. Then (with $\hat{u}$ and $\hat{\bar{f}}$ denoting the respective transforms of u and $\bar{f}$) we find

$$-A(\xi,\xi)\hat{u} - \sum_{j,k=1}^{n} b_{jk}\xi_k \frac{\partial \hat{u}}{\partial \xi_j} = \hat{\bar{f}}.$$

This is a first order partial differential equation that can be solved by the method of characteristics for given noncharacteristic initial conditions. The characteristic curves are determined by (B' denotes B transpose)

$$\frac{d\xi}{d\tau} = B'\xi$$

and $\hat{u}$ must satisfy

$$\frac{d\hat{u}}{d\tau} = A(\xi,\xi) - \hat{\bar{f}}.$$

Here τ denotes the parameter along the characteristic curves.

Hence the linear feedback in which the B matrix is altered simply changes the characteristic curves used in solving for $\hat{u}$. On examining Hörmander's work [H], there is a quadratic form whose positive definite matrix is the controllability matrix of the associated linear system $\dot{y} = By + Av$.

III. Simplifying Coefficients

We now study the problem of moving the partial differential equation (5)

$$\sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} + C(x)u = f$$

to the hypoelliptic (rank $(A,BA,B^2A,\ldots,B^{n-1}A)=n$) Kolmogorov equation (6)

$$\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial y_j \partial y_k} + \sum_{j,k=1}^{n} b_{jk}\, y_j \frac{\partial u}{\partial y_k} = \bar{f}.$$

Again the matrices $A = (a_{jk})$ and $(A_{jk}(x))$ are assumed to be symmetric, positive semidefinite, and have constant rank m. In Hörmander's vector field notation we must transform (10)

$$\sum_{j=1}^{m} X_j^2 u + X_0 u + C(x)u = f,$$

to the hypoelliptic equation (9)

$$\sum_{j=1}^{m} Y_j^2\, u + Y_0 u = \bar{f},$$

where both sets $\{X_1, X_2, \ldots, X_m\}$ and $\{Y_1, Y_2, \ldots, Y_m\}$ are linearly independent.

A few necessary definitions are in order.

If X and Y are C^{∞} vector fields on $\mathbb{R}^n$, then the <u>Lie bracket</u> of X and Y is

$$[X,Y] = -\frac{\partial Y}{\partial x} X + \frac{\partial X}{\partial x} Y,$$

where $\frac{\partial Y}{\partial x}$ and $\frac{\partial X}{\partial x}$ are Jacobian matrices, x being the variable for $\mathbb{R}^n$.

Successive Lie brackets such as $\Big[X,[X,Y]\Big]$, $\Big[Y,\ [X,Y]\Big]$, $\Big[\Big[X,\ [X,Y]\Big],Y\Big]$, etc. can be taken. A standard notation is

$$(\mathrm{ad}^0 X, Y) = Y$$

$$(\mathrm{ad}^1 X, Y) = [X, Y]$$

$$(\mathrm{ad}^2 X, Y) = \Big[X, [X, Y]\Big]$$

$$\vdots$$

$$(\mathrm{ad}^j X, Y) = [X, (\mathrm{ad}^{j-1} X, Y)]$$

We let $\langle \cdot , \cdot \rangle$ denote the dual product of one forms and vector fields. Given a C^∞ function h on R^n we define the Lie derivative of h with respect to the vector field X as

$$L_X h = \langle dh, f \rangle.$$

Successive Lie derivatives are

$$\overset{0}{L_X} h = h$$

$$\overset{1}{L_X} h = L_X h$$

$$\overset{2}{L_X} h = L_X L_X h$$

$$\vdots$$

$$L_X^j h = L_X L_X^{j-1} h.$$

Moreover, the Lie derivative of the one form dh with respect to X is

$$L_X(dh) = \left[\frac{\partial (dh)^*}{dx} X\right]^* + (dh)\frac{\partial X}{\partial x} \, ,$$

where $*$ denotes transpose.

The three types of Lie derivatives satisfy the formula

(18) $$L_X \langle dh, Y \rangle = \langle L_X(dh), Y \rangle - \langle dh, [X, Y] \rangle.$$

We begin our discussion with an example.

Example 3.1. Consider the partial differential equation

$$x_3^2\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 2\,\frac{\partial^2 u}{\partial x_1\partial x_2}\right] + 2x_3\left[\frac{\partial^2 u}{\partial x_1\partial x_3} + \frac{\partial^2 u}{\partial x_2\partial x_3}\right] + \frac{\partial^2 u}{\partial x_3^2}$$

$$+ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \left\{\left[x_1 - \frac{x_3^2}{2}\right]^2 + \left[x_2 - \frac{x_3^2}{2}\right] + x_3\left[\left[x_1 - \frac{x_3^2}{2}\right]^3 + \left[x_2 - \frac{x_3^2}{2}\right]^3\right]\right\}\frac{\partial u}{\partial x_1}$$

(19)

$$+ \left\{x_3 + \sin\left[x_2 - \frac{x_3^2}{2}\right] + x_3\left[\left[x_1 - \frac{x_3^2}{2}\right]^3 + \left[x_2 - \frac{x_3^2}{2}\right]^3\right]\right\}\frac{\partial u}{\partial x_2}$$

$$+ \left[\left[x_1 - \frac{x_3^2}{2}\right]^3 + \left[x_2 - \frac{x_3^2}{2}\right]^3\right]\frac{\partial u}{\partial x_3} = \tilde{f}.$$

We write this in a vector field notation

(20) $$X_1^2 + X_0 = \tilde{f}, \quad \text{where}$$

$$X_0 = \begin{bmatrix} \left[x_1 - \frac{x_3^2}{2}\right]^2 + \left[x_2 - \frac{x_3^2}{2}\right] + x_3\left[\left[x_1 - \frac{x_3^2}{2}\right]^3 + \left[x_2 - \frac{x_3^2}{2}\right]^3\right] \\ x_3 + \sin\left[x_2 - \frac{x_3^2}{2}\right] + x_3\left[\left[x_1 - \frac{x_3^2}{2}\right]^3 + \left[x_2 - \frac{x_3^2}{2}\right]^3\right] \\ \left[x_1 - \frac{x_3^2}{2}\right]^3 + \left[x_2 - \frac{x_3^2}{2}\right]^3 \end{bmatrix} \text{and } X_1 = \begin{bmatrix} x_3 \\ x_3 \\ 1 \end{bmatrix}$$

The coordinate change

(21)
$$s_1 = x_1 - \frac{x_3^2}{2}$$
$$s_2 = x_2 - \frac{x_3^2}{2}$$
$$s_3 = x_3$$

yields

(22) $$S_1^2 + S_0 = \tilde{f}.$$

where $$S_0 = \begin{bmatrix} s_2 + s_1^2 \\ s_3 + \sin s_2 \\ s_1 + s_2^3 \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

i.e. we have the partial differential equation

$$(23) \quad \frac{\partial^2 u}{\partial s_3^2} + \left[s_2 + s_1^2\right] \frac{\partial u}{\partial s_1} + \left[s_3 + \sin s_2\right] \frac{\partial u}{\partial s_2} + \left[s_1^3 + s_2^3\right] \frac{\partial u}{\partial s_3} = \tilde{f}$$

The coordinate change

$$(24) \quad \begin{aligned} y_1 &= s_1 \\ y_2 &= s_2 + s_1^2 \\ y_3 &= s_3 + \sin s_2 + 2s_1\left[s_2 + s_1^2\right] \end{aligned}$$

gives

$$(25) \quad Y_1^2 + Y_0 = \tilde{f}, \quad \text{where}$$

$$(26) \quad Y_0 = \begin{bmatrix} y_2 \\ y_3 \\ y_1^3 + \left[y_2 - y_1^2\right]^3 + \left[\cos\left[y_2 - y_1^2\right]\right]\left[y_3 + 2y_1y_2\right] + 2y_2^2 + 2y_1y_3 \end{bmatrix}$$

and $Y_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Thus we obtain the partial differential equation

$$(27) \quad \frac{\partial^2 u}{\partial y_3^2} + y_2 \frac{\partial u}{\partial y_1} + y_3 \frac{\partial u}{\partial y_2} +$$

$$\left[y_1^3 + \left[y_2 - y_1^2\right]^3 + \left[\cos\left[y_2 - y_1^2\right]\right]\left[y_3 + 2y_1y_2\right] + 2y_2^2 + 2y_1y_3\right] \frac{\partial u}{\partial y_3} = \tilde{f}.$$

Replacing $\tilde{f} - \left[\text{term involving } \frac{\partial u}{\partial y_3}\right]$ by $\bar{f}$ we have

(28) $$\frac{\partial^2 u}{\partial y_3^2} + y_2 \frac{\partial u}{\partial y_1} + y_3 \frac{\partial u}{\partial y_2} = \bar{f}.$$

We compute interesting Lie brackets of the vector fields S_0 and S_1 from equation (22). The Lie brackets

$$S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \; [S_0,S_1] = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \text{ and } [S_0,[S_0,S_1]] = \begin{bmatrix} 1 \\ \cos s_2 \\ 3s_2^2 \end{bmatrix}$$

span $\mathbb{R}^3$ near the origin. Moreover

$[[S_0,[S_0,S_1]],S_1] \equiv 0$, $[[S_0,S_1],S_1] \equiv 0$ and

$$[[S_0,[S_0,S_1]],[S_0,S_1]] = \begin{bmatrix} 0 \\ \cos s_2 \\ 6s_2 \end{bmatrix}$$

This last Lie bracket is in the span of S_1 and $[S_0,S_1]$. Similar Lie bracket relations hold if we compute in the original x_1,x_2,x_3 coordinates since Lie brackets are preserved under coordinate transformations.

We examine equation (10)

$$\sum_{j=1}^{m} X_j^2 u + X_0 u + C(x)u = f,$$

and replace $f - C(x)u$ by $\tilde{f}$ to obtain

$$\sum_{j=1}^{m} X_j^2 u + X_0 u = \tilde{f}. \tag{29}$$

Taking equation (10) to the Kolmogorov equation (9) is now identical to moving equation (29) to equation (9).

Definition 3.1. A feedback transformation on the second order partial differential equation (29) consists of

i) nonsingular coordinate changes on $\mathbb{R}^n$ (taking the origin to the origin)

ii) replacing $f - \sum_{j=1}^{m} \alpha_j(x) X_j u$ by $\bar{f}$, where the $\alpha_j(x)$ are smooth functions.

We remark that ii) allows the feedback of linear combinations of vector fields $X_1, X_2, \ldots, X_m$ applied to u. The principal symbol and hypoellipticity of equation (29) are invariant under a feedback transformation.

Definition 3.2. Two second order partial differential equations of the form (10) (or (29)) are feedback equivalent if there is a feedback transformation taking one to the other.

For our hypoelliptic Kolmogorov equation (9) we assume Kolmogorov indices $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_m$. With respect to equation (29) we define the following sets of vector fields. Let

$$\begin{aligned}
W = & \Big\{X_1, [X_0,X_1], \ldots, (\mathrm{ad}^{\ell_1-1} X_0, X_1), X_2, [X_0,X_2], \ldots, \\
& (\mathrm{ad}^{\ell_2-1} X_0, X_2), \ldots, X_m, [X_0,X_m], \ldots, (\mathrm{ad}^{\ell_{m}-1} X_0, X_m)\Big\}, \\
W_i = & \Big\{X_1, [X_0,X_1], \ldots, (\mathrm{ad}^{i-1} X_0, X_1), X_2, [X_0,X_2], \ldots, (\mathrm{ad}^{i-1} X_0, X_2), \\
& \ldots, X_m, [X_0,X_m], \ldots, (\mathrm{ad}^{i-1} X_0, X_m)\Big\}, \quad i=1,2,\ldots,\ell_1-1, \\
W_0 = & \{0\}.
\end{aligned}$$

We now state and prove our main result and remark that our feedback equivalence is local (in a neighborhood of the origin).

Theorem 3.1. The linear second order partial differential equation (29)

$$\sum_{j=1}^{m} X_j^2 u + X_0 u = \tilde{f},$$

with $X_0(0) = 0$, is feedback equivalent to the Kolmogorov partial differential equation (9)

$$\sum_{j=1}^{m} Y_j^2 u + Y_0 u = \bar{f},$$

having Kolmogorov indices $\ell_1 \geq \ell_2 \geq \dots \geq \ell_m$ if and only if

1) the vector fields in W are linearly independent and span $W_i = \text{span } W_i \cap W$, $i=1,2,\dots,\ell_1-1$, and
2) for each $i=1,2,\dots,\ell_1-1$, the Lie bracket of any two vector fields in W_i is a linear combination of vector fields in W_{i-1}.

Proof. The proof for a general m is a straightforward generalization of the proof for $m = 2$, so we consider this case only. Then

$$W = \left\{X_1, [X_0,X_1], \dots, (ad^{\ell_1-1}X_0,X_1), X_2, [X_0,X_2], \dots, (ad^{\ell_2-1}X_0,X_2)\right\}$$

and we set $k = \ell_1 - \ell_2$. We begin by assuming conditions 1) and 2).

We first apply a nonsingular change of coordinates that is standard for transformation theory of nonlinear ordinary differential equations control systems [JR],[HSM],[SH],[K3].

Solve in order the system of ordinary differential equations with the initial conditions indicated:

$$\frac{dx}{ds_1} = (ad^{\ell_1-1}X_0,X_1), \quad x(0,0,\dots,0) = 0$$

$$\frac{dx}{ds_2} = (ad^{\ell_1-2}X_0,X_1), \quad x(s_1,0,0,\dots,0) = x(s_1)$$

$$\vdots$$

$$\frac{dx}{ds_{k+1}} = (ad^{\ell_2-1}X_0,X_1), \quad x(s_1,s_2,\dots,s_k,0,0,\dots,0) = x(s_1,s_2,\dots,s_k)$$

$$\frac{dx}{ds_{k+2}} = (ad^{\ell_2-1}X_0,X_2), \quad x(s_1,s_2,\dots,s_{k+1},0,0,\dots,0) = x(s_1,s_2,\dots,s_{k+1})$$

$$\frac{dx}{ds_{k+3}} = (ad^{\ell_2-2}X_0,X_1), \quad x(s_1,s_2,\dots,s_{k+2},0,0,\dots,0) = x(s_1,s_2,\dots,s_{k+2})$$

$$\frac{dx}{ds_{k+4}} = (ad^{\ell_2-2}X_0,X_2), \quad x(s_1,s_2,\dots,s_{k+3},0,0,\dots,0) = x(s_1,s_2,\dots,s_{k+3})$$

$$\vdots$$

$$\frac{dx}{ds_{n-1}} = X_1, \quad x(s_1,s_2,\dots,s_{n-2},0,0,) = x(s_1,s_2,\dots,s_{n-2})$$

$$\frac{dx}{ds_n} = X_2, \quad x(s_1,s_2,\dots,s_{n-1},0) = x(s_1,s_2,\dots,s_{n-1})$$

Since the set W of vector fields spans $\mathbb{R}^n$, we can locally invert to determine our new coordinates $s_1,s_2,\dots,s_n$ for $\mathbb{R}^n$.

We define manifolds

$$S_\ell = \left\{s = (s_1,s_2,\dots,s_n) \in \mathbb{R}^n : s_i = 0, \ \ell + 1 \le i \le n\right\} \tag{31}$$

and note that

$$X_2 = \frac{\partial}{\partial s_n}, \quad X_1\Big|_{S_{n-1}} = \frac{\partial}{\partial s_{n-1}}, \quad [X_0,X_2]\Big|_{S_{n-2}} = \frac{\partial}{\partial s_{n-2}}, \quad [X_0,X_1]\Big|_{S_{n-3}} = \frac{\partial}{\partial s_{n-3}},$$

$$\ldots,\ (ad^{\ell_2-2}X_0,X_2)\Big|_{S_{k+4}} = \frac{\partial}{\partial s_{k+4}},\ (ad^{\ell_2-2}X_0,X_1)\Big|_{S_{k+3}} = \frac{\partial}{\partial s_{k+3}},$$

$$(32)\qquad (ad^{\ell_2-1}X_0,X_2)\Big|_{S_{k+2}} = \frac{\partial}{\partial s_{k+2}},\ (ad^{\ell_2-1}X_0,X_1)\Big|_{S_{k+1}} = \frac{\partial}{\partial s_{k+1}},\ \ldots,$$

$$(ad^{\ell_1-2}X_0,X_2)\Big|_{S_2} = \frac{\partial}{\partial s_2},\ (ad^{\ell_1-1}X_0,X_1)\Big|_{S_1} = \frac{\partial}{\partial s_1}$$

From assumption 1) on W and our choice of coordinates as noted in (30) and (32) we obtain

$$(33)\quad X_0 = \begin{bmatrix} X_{01} \\ X_{02} \\ \vdots \\ X_{0k} \\ X_{0k+1} \\ X_{0k+2} \\ X_{0k+3} \\ \vdots \\ X_{0n-3} \\ X_{0n-2} \\ X_{0n-1} \\ X_{0n} \end{bmatrix} \qquad \begin{bmatrix} s_2 + X'_{01}(s) \\ s_3 + X'_{02}(s) \\ \vdots \\ s_{k+1} + X'_{0k}(s) \\ s_{k+3} + X'_{0k+1}(s) \\ s_{k+4} + X'_{0k+2}(s) \\ s_{k+5} + X'_{0k+3}(s) \\ \vdots \\ s_{n-1} + X'_{0n-3}(s) \\ s_n + X'_{0n-2}(s) \\ X'_{0n-1}(s) \\ X'_{0n}(s) \end{bmatrix}$$

Here $X'_{01}(s)$, $X'_{02}(s),\ldots,X'_{0k-1}(s)$ are not linear in $s_2,s_3,\ldots,s_n$, and $X'_{0k}(s),X'_{0k+1}(s),\ldots,X'_{0n}(s)$ are not linear in $s_2,s_3,\ldots,s_{k+1},s_{k+3},s_{k+4},\ldots,s_n$. Equation (33) can also be derived by simply considering the linear part of X_0 in the s coordinates.

From assumption 2)

$$[X_1, X_2] = \frac{\partial X_1}{\partial s_n} = 0$$

This together with $X_1 \Big|_{S_{n-1}} = \frac{\partial}{\partial s_{n-1}}$ imply $X_1 = \frac{\partial}{\partial s_{n-1}}$

Since the Lie brackets of vector fields in

$$W_2 = \left\{X_1, [X_0, X_1], X_2, [X_0, X_2]\right\}$$

are in $W_1 = \left\{X_1, X_2\right\}$, and (at the origin)

$$[[X_0, X_2], X_2] = \frac{\partial^2 X_0}{\partial s_n^2}$$

$$[[X_0, X_1], X_2] = \frac{\partial^2 X_0}{\partial s_{n-1} \partial s_n}$$

$$[[X_0, X_2], X_1] = \frac{\partial^2 X_0}{\partial s_n \partial s_{n-1}}$$

$$[[X_0, X_1], X_1] = \frac{\partial^2 X_0}{\partial s_{n-1}^2}$$

we find that $X'_{01}(s)$, $X'_{02}(s), \ldots, X'_{0n-2}(s)$ in (33) have no terms containing $s^2_{n-1}, s_{n-1}s_n, s_n^2$ in their formal power series expansion about 0.

Since the Lie brackets of vector fields in are in $W_3 = \left\{X_1, [X_0, X_1], X_2, [X_0, X_2], (\mathrm{ad}^2 X_0, X_1), (\mathrm{ad}^2 X_0, X_2)\right\}$

$W_2 = \left\{X_1, [X_0, X_1], X_2, [X_0, X_2]\right\}$ and (at the origin)

$$[(\mathrm{ad}^2 X_0, X_2), X_2] = \frac{\partial^2 X_0}{\partial s_{n-2} \partial s_n}$$

$$[(\mathrm{ad}^2 X_0, X_2), X_1] = \frac{\partial^2 X_0}{\partial s_{n-2} \partial s_{n-1}}$$

$$[(\mathrm{ad}^2 X_0, X_1), X_2] = \frac{\partial^2 X_0}{\partial s_{n-3} \partial s_n}$$

$$[(\mathrm{ad}^2 X_0, X_1), X_1] = \frac{\partial^2 X_0}{\partial s_{n-3} \partial s_{n-1}}$$

$$[(\mathrm{ad}^2 X_0, X_2), [X_0, X_2]] = \frac{\partial^2 X_0}{\partial s_{n-2}^2}$$

$$[(\mathrm{ad}^2 X_0, X_2), [X_0, X_1]] = \frac{\partial^2 X_0}{\partial s_{n-2} \partial s_{n-3}}$$

$$[(\mathrm{ad}^2 X_0, X_1), [X_0, X_2]] = \frac{\partial^2 X_0}{\partial s_{n-3} \partial s_{n-2}}$$

$$[(\mathrm{ad}^2 X_0, X_1), [X_0, X_1]] = \frac{\partial^2 X_0}{\partial s_{n-3}^2}$$

we find that $X'_{01}(s), X'_{02}(s), \ldots, X'_{0n-4}(s)$ have no terms containing

$s_{n-3}^2, s_{n-3}s_{n-2}, s_{n-3}s_{n-1}, s_{n-3}s_n, s_{n-2}^2, s_{n-2}s_{n-1}, s_{n-2}s_n$ in their formal power series expansions about 0.

Continuing in this manner we find that the vector field X_0 from (33) becomes (with the definitions of $\ddot{X}_0$ being obvious)

$$(34)\quad X_o = \begin{bmatrix} s_2 + \ddot{X}_{01}(s_1) \\ s_3 + \ddot{X}_{02}(s_1, s_2) \\ \cdot \\ \cdot \\ \cdot \\ s_{k+1} + \ddot{X}_{0k}(s_1, s_2, \ldots, s_k) + c_{k,k+2}s_{k+2} \\ s_{k+3} + \ddot{X}_{0k+1}(s_1, s_2, \ldots, s_{k+1}, s_{k+2}) + c_{k+1,k+2}s_{k+2} \\ s_{k+4} + \ddot{X}_{0k+2}(s_1, s_2, \ldots, s_{k+1}, s_{k+2}) + c_{k+2,k+2}s_{k+2} \\ s_{k+5} + \ddot{X}_{0k+3}(s_1, s_2, \ldots, s_{k+3}, s_{k+4}) + c_{k+3,k+2}s_{k+2} \\ \cdot \\ \cdot \\ \cdot \\ s_{n-1} + \ddot{X}_{0n-3}(s_1, s_2, \ldots, s_{n-3}, s_{n-2}) \\ s_n + \ddot{X}_{0n-2}(s_1, s_2, \ldots, s_{n-3}, s_{n-2}) \\ \ddot{X}_{0n-1}(s) \\ \ddot{X}_{0n}(s) \end{bmatrix}$$

where the c's are constants .

Because we can work backwards from this point in the proof, we have that X_0 can be represented as in equation (34) and $X_1 = \frac{\partial}{\partial s_{n-1}}$, $X_2 = \frac{\partial}{\partial s_{n-2}}$ if and only if conditions 1) and 2) in the statement of the theorem hold.

Next we define the desired y coordinates. Let

$$y_1 = s_1$$

$$y_2 = L_{X_0} y_1 = \langle dy_1, X_0 \rangle$$

$$y_3 = L_{X_0} y_2 = \langle dy_2, X_0 \rangle$$

$$\vdots$$

(35) $$y_{\ell_1} = L_{X_0} y_{\ell_1 - 1} = \langle dy_{\ell_1 - 1}, X_0 \rangle$$

$$y_{\ell_1 + 1} = s_{k+2}$$

$$y_{\ell_1 + 2} = L_{X_0} y_{\ell_1 + 1} = \langle dy_{\ell_1 + 1}, X_0 \rangle$$

$$y_{\ell_1 + 3} = L_{X_0} y_{\ell_1 + 2} = \langle dy_{\ell_1 + 2}, X_0 \rangle$$

$$\vdots$$

$$y_n = L_{X_0} y_{n-1} = \langle dy_{n-1}, X_0 \rangle$$

It is shown in [HSM] that this coordinate change is nonsingular.

The result of this coordinate change on the vector field X_0 is

$$L_{X_0} y_1 \frac{\partial}{\partial y_1} + L_{X_0} y_2 \frac{\partial}{\partial y_2} + \ldots + L_{X_0} y_{\ell_1 - 1} \frac{\partial}{\partial y_{\ell_1 - 1}} + L_{X_0} y_{\ell_1} \frac{\partial}{\partial y_{\ell_1}} +$$

$$L_{X_0} y_{\ell_1 + 1} \frac{\partial}{\partial y_{\ell_1 + 1}} + L_{X_0} y_{\ell_1 + 2} \frac{\partial}{\partial y_{\ell_1 + 2}} + \ldots + L_{X_0} y_{n-1} \frac{\partial}{\partial y_{n-1}} + L_{X_0} y_n \frac{\partial}{\partial y_n} =$$

$$y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} + \ldots + y_{\ell_1} \frac{\partial}{\partial y_{\ell_1 - 1}} + L_{X_0} y_{\ell_1} \frac{\partial}{\partial y_{\ell_1}} +$$

(36)

$$y_{\ell_1 + 2} \frac{\partial}{\partial y_{\ell_1 + 1}} + y_{\ell_1 + 3} \frac{\partial}{\partial y_{\ell_1 + 2}} + \ldots + y_n \frac{\partial}{\partial y_{n-1}} + L_{X_0} y_n \frac{\partial}{\partial y_n}.$$

From the form of X_0 in (33) and (34) and the definition of the y coordinates (35), we must have

$$\langle dy_i, X_1\rangle = \langle dy_i, \begin{bmatrix}0\\0\\ \vdots\\0\\1\\0\end{bmatrix}\rangle = 0, \quad i \neq \ell_1$$

$$\langle dy_{\ell_1}, X_1\rangle = \langle dy_{\ell_1}, \begin{bmatrix}0\\0\\ \vdots\\0\\1\\0\end{bmatrix}\rangle = 1,$$

(37) $$\langle dy_i, X_2\rangle = \langle dy_i, \begin{bmatrix}0\\0\\ \vdots\\0\\0\\1\end{bmatrix}\rangle = 0, \quad i \neq \ell_1, n$$

$$\langle dy_{\ell_1}, X_2\rangle = \langle dy_{\ell_1}, \begin{bmatrix}0\\0\\ \vdots\\0\\0\\1\end{bmatrix}\rangle = d, \quad \text{a constant}$$

$$\langle dy_n, X_2\rangle = \langle dy_n, \begin{bmatrix}0\\0\\ \vdots\\0\\0\\1\end{bmatrix}\rangle = 1.$$

To see this we show the terms from (33) and (35) involving only the pertinent higher subscripted linear s_i (and we let c denote appropriate constants from (34) and c can change as needed):

$$y_1 = s_1$$

$$y_2 = L_{X_0} y_1 = s_2 + \dots$$

$$y_3 = L_{X_0} y_2 = s_3 + \dots$$

$$\vdots$$

$$y_k = L_{X_0} y_{k-1} = s_k + \cdots$$

$$y_{k+1} = L_{X_0} y_k = s_{k+1} + c\, s_{k+2} + \cdots$$

$$y_{k+2} = L_{X_0} y_{k+1} = s_{k+3} + c\, s_{k+4} + \cdots$$

$$\vdots$$

$$y_{\ell_1} = L_{X_0} y_{\ell_1 - 1} = s_{n-1} + cs_n + \cdots = s_{n-1} + ds_n + \cdots$$

$$y_{\ell_1+1} = s_{k+2}$$

$$y_{\ell_1+2} = L_{X_0} y_{\ell_1+1} = s_{k+4} + \cdots$$

$$y_{\ell_1+3} = L_{X_0} y_{\ell_1+2} = s_{k+6} + \cdots$$

$$\vdots$$

$$y_n = L_{X_0} y_{n-1} = s_n + \cdots \quad .$$

Hence in the y coordinates X_1 and X_2 become

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

respectively. We assume that equation (29) is now in our y coordinates. Replacing $\tilde{f} - \left[L_{X_0} y_{\ell_1} \frac{\partial u}{\partial y_{\ell_1}} + L_{X_0} y_n \frac{\partial u}{\partial y_n}\right]$ by

$\bar{f}$ in (29) and letting Y_0 be the vector field defined by (36) (with $\left[L_{X_0} y_{\ell_1} \frac{\partial u}{\partial y_{\ell_1}}\right.$ and $L_{X_0} y_n \frac{\partial}{\partial y_n}$ omitted$\Big]$, we have equation (9)

$$\sum_{j=1}^{m} Y_j^2\, u + Y_0 u = \bar{f}, \qquad \square$$

We remark that Theorem 3.1 is an application to partial differential equations of multi-input extensions of the work of Brockett [B] on nonlinear control systems of ordinary differential equations.

IV Other Considerations

Suppose that our hypoelliptic partial differential equation (5)

$$\sum_{j,k=1}^{n} A_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} + C(x)u = f$$

is elliptic with $(A_{jk}(x))$ being positive definite. Referring to the notation used in Theorem 3.1 $m=n$, $\ell_1 = \ell_2 = \ldots = \ell_n = 1$, and $W = \left\{X_1, X_2, \ldots, X_n\right\}$. If the hypotheses of that theorem are assumed, then we can use feedback to eliminate all first order derivatives (as well as $C(x)u$). Hence we obtain a second order constant coefficient equation involving no first order or constant terms.

We have considered the possible feedback of first order spatial derivatives of the solution in the directions of the vector fields whose squares contain the principal part of the operator. If time t is added as in equation (4), then we could also consider the possible feedback of the first order time derivative of u to change damping properties.

In practical applications, a physical system modeled by a partial differential equation (or system of partial differential equations) has sensor and control actuators that act at a finite number of points on the system. This leads to the application of finite element (or finite difference) methods for control, as is well noted in the literature. Perhaps simplifying the partial differential equations as in this paper can lead to an easier finite element approach.

A recent paper by Juang and Horta [JH] on the control of a cantilever beam uses a finite element method so that a finite number of shape functions are to be controlled. Interestingly, strain gauges are used as sensors, and this information is fed back in the control action. The strain gauges are essentially applied to allow for computations of second order spatial derivatives. Since the beam equation is fourth order, this feedback does not disrupt the principal symbol of the partial differential equation.

We have considered second order partial differential equations, but higher even ordered equations are of interest since the hypoellipticity theory has moved in that direction.

Hörmander [H] shows that the loss of hypoellipticity can lead to discontinuous solutions by a simple application of the Frobenius theorem. The book on nonlinear elastic deformations by Ogden [O] refers to papers by Knowles and Sternberg [KS1], [KS2] in which the loss of ellipticity implies the emergence of discontinuous solutions. It would be interesting to compare these mathematical and physical phenomena, particularly from a geometric viewpoint.

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